# GAME THEORY AND THEORY OF HYDRAULIC NETWORKS FOR THE ANALYSIS OF A PROBLEM OF THE MOVEMENT OF TRAFFIC FLOWS IN THE URBANIZED TERRITORIES

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**Summary**

We consider a problem of distribution of traffic flows in the urbanised territories. By means of methods of game theory and mathematical methods of the theory of hydraulic networks we set a task of search of equilibrium state of the movement of city streams. We apply methods of cyclic coordination of the theory of hydraulic networks to the solution of this task.

**Keywords:** the urbanized territories, distribution of the movement of flows in networks, game theory, theory of hydraulic networks, balance of Nash, evacuation of the population

## Introduction

The problems of the engineering systems in urban areas, and in particular, road transport systems are well known [1]. Every inhabitant of a city experiences them daily. Both on weekdays and weekends, many cities are practically in traffic jams and the social discontent of the population, the enormous economic costs and the significant environmental degradation are the outcome. And it is impossible to solve these problems only by managing traffic flows. This is a complex problem. And first of all, the problem of incorrect disposal and location of neighbourhoods, industrial areas, stores, etc. The structure makes people travel. But for the correct design, for integrated evaluation, transport models are also necessary [2,3]. The models and methods proposed are based on the methods of game theory [4] and models of hydraulic network theory [5], which are widely used.

## 1 Descriptive model

A city can be represented as a multitude of objects (neighbourhoods, colonies, barracks, individual houses, businesses, stores, points of entry / exit of the city, etc.), among which transport moves, transporting various types of cargo and people. Among these objects are highways that consist of sections of streets between intersections, points of entry and exit of the objects listed along which the traffic is carried out. We will assume that all the objects are linked to points (places) of entry and exit to the highway from the neighbourhoods; Through these sites, neighbourhoods allow the entry and exit of vehicles; see the example in fig. one.

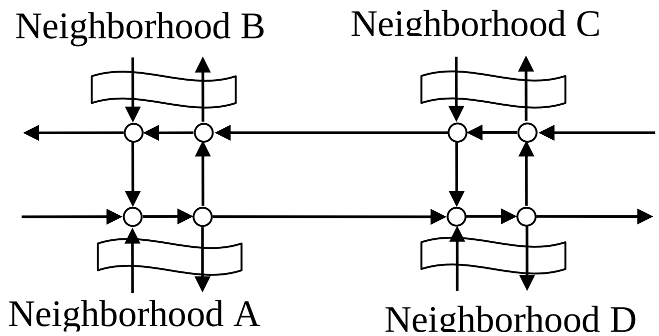


*Figure 1- Entry-exit from neighbourhoods*

Arrows indicate one-way lanes; circles indicate points (places) of the input-output flow, merger-separation flow. At intersections, flows merge and separate. A fairly general intersection is shown in fig. 2.

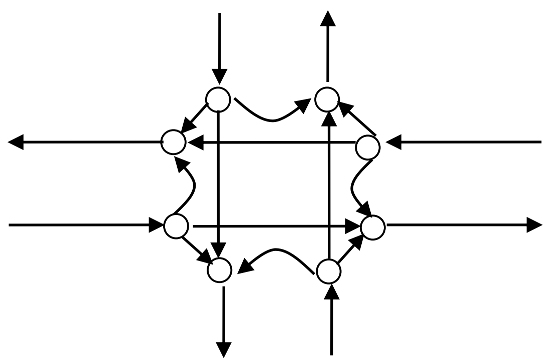

*Figure 2 – Example of an intersection.*

Traffic flows along the roads between the various points mentioned above. Therefore, the structure of the movement can be described as a directed graphic. The top of the graph - the entry point - leaves the intersection, entry - exit neighbourhood, entry - exit to the city. The arcs of the graph are sections of the road between the peaks. The direction of the arrow indicates the direction of flow.

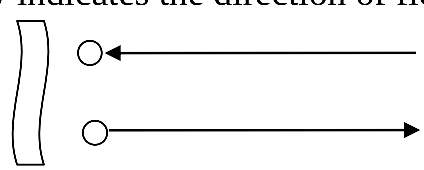

*Figure 3 – Entrance - exit of the city*

The vertices from which the flow enters the network will be called sources; the vertices to which the flow comes from the network will be called drainages.

We assume that the flow consists of individual streams, characterised by a single trajectory of movement towards the final vertex: the drainage. The current transport units are the same and move uniformly, the movement speed of these units in different arcs can be different. To what point can the flow be a column in movement, organised, infinite? The complete flow consists of a set of individual streams, the total value of the flows generated from the vertex *i* to the vertex *j* is denoted by *Qij*. The paths of the streams that enter *Qij* are, in general, different. We assume that each stream is controlled by a subject that organises the movement of this flow.

The set of all these subjects form the set of players *I*. For each player $\gamma \in I$, the strategy $\xi_\gamma$ is the route from source *i* to sink *j* from the set of all strategies - routes $\chi_\gamma$, connecting these vertices. The criterion of each subject is the movement time from *i* to *j*, its objective is to minimize the movement time of each transport unit and, therefore, all the current transport units. The value of this criterion is influenced by other players whose path intersects with the player's path $\gamma$. In superimposed areas, they increase the density of flow, which reduces the speed of movement and, consequently, increases the time. As a result, we get the game in a normal way:

$$G = \left\langle I;\ \ \chi_\gamma, \gamma \in I;\ \ \ \varphi_\gamma(\xi_1, \xi_2, \xi_3, \ldots \xi_\gamma, \ldots, \xi_{|I|}) \underset{\xi_\gamma \in \chi_\gamma}{\Longrightarrow} \min, (\xi_1, \xi_2, \xi_3, \ldots \xi_\gamma, \ldots, \xi_{|I|}) \in \chi = \prod_{k \in I} \chi_k, \gamma \in I \right\rangle$$

the equilibrium in this game, we will be understand it with the help of the Nash equilibrium [4].

In the sequel, we will assume that the set of players *I* is infinite, it is continuous. Accordingly, the *Qij* flows will be divided into streams $x_{ij}^\gamma$ of arbitrarily small magnitude.

## 2 Analysis of the movement of the flow along the arc

Obviously, for the movement time all along path to be minimal, it is necessary that the speed of movement of each participant in the arc is maximum. But the value of the speed of this movement of participants involuntarily has a negative impact on other participants. In their choice of movement parameters, they also strive to maximize speed. The increase in flow leads to a decrease in the speed of movement of the driver in question, which leads to an increase in their time.

Consider the movement of a single arc, so we omit all indexes related to the arc. We present the following notation: *L* is the length of the network section, *T* is the travel time in the section, *x* is the number of cars that passed through the section of the road per unit of time, $\rho$ is the density flow is the number of cars per unit length in a lane, *s* is the number of lanes on the road, *w* is the speed at which the flow moves, $\lambda$ is the average length of the section by 1 automobile in the first lane.

According to the definition of density, $\rho = 1 / \lambda$.. Let *w* be the speed of the car, $w_{max}$ be the maximum speed. The time it takes a car to travel a length of $\lambda$ is $\tau = \lambda / v$. The number of cars per unit of time will be equal to $\kappa = 1 / \tau$. From this definition we get $x = \kappa s = \frac{1}{\tau} s = \frac{v}{\lambda} = w\rho s$ . We assume that the velocity and density of the flow depend linearly on each other if $w/w_{max} + \rho/\rho_{max} = 1$ (Greenshield formula), therefore $w = w_{max}(1 - \rho/\rho_{max})$, or $\rho = \rho_{max}(1 - w/w_{max})$. We replace it in the flow, and we get

$x = s\, w\, \rho_{max}(1 - w/w_{max})$. The resulting function is a parabola with the branches down, the maximum is reached in $w = w_{\max}/2$, and respectively $x_{max} = s(w_{max}\ \rho_{max})/4$. Therefore, we obtained the maximum flow value that can be passed along the road.

Substitute instead of $\rho$ its expression, we obtain $w^2 - w_{max} w + (w_{max}/(s\rho_{max}))x = 0$. According to the Vieta formulas that we obtain, taking into account the fact that everyone wants to maximize their speed $w = w_{max}\left(1 + \sqrt{1 - x/x_{max}}\right)/2$. From this, we obtain that the movement time along a network section is expressed by the following relation: $\tau(x) = 2\tau_{min}/\left(1 + \sqrt{1 - x/x_{max}}\right)$, where $\tau_{min}$ is the minimum time movement along a section in the case when the flow through it is zero. For clarity, the form of this function will give a graph of the function $\tau(x) = 1/\left(1 + \sqrt{1 - x}\right)$.

## 3 The concept of a layer, the equilibrium relations of a layer, and the invariable transformation of layer flows

### 3.1 The concept of the layer balance relationship

Let $G = \langle E, V, H \rangle$ be a directed graph, $E$ and $V$ be finite sets, $H$ is an assignment $H{:}V \to E \times E$. The elements of $E$ are called vertices of a graph; the elements of $V$ are arcs. For each arc $v \in V$, in the representation $H(v) = (h1(v), h2(v))$, $h1(v)$ is the beginning of the arc $v$, and $h2(v)$ is the ending. Denote that $V^{+}(i) = \{v \in V \mid h2(v) = i\}$ is the set of arcs that enter the vertex $i$, and $V^{-}(i) = \{v \in V \mid h1(v) = i\}$ is the set of arcs that come out of the vertex $i$.

For each pair $(i, j)$ of the vertices, the numbers $Qij$ are given, specifying the flow from the vertex - source $i$ to the vertex - to the discharging $j$. These streams are broken down into separate streams and distributed through the network; as a result, for each arc $v \in V$, we obtain $q_{i\,j}^{v}$, the stream along the arc $v$, moved from the source $i$ to the ending $j$.

### 3.2 Divide traffic flows into layers.

Take the vertex $i_0$, which is the source of the flow to other vertices. The flows that enter the vertices of $i \in E$ are indicated with $q_i\ (i_0)$, for the vertex $i_0 \in E$ it will be $q_{i_0}(i_0)$. The set $S(i_0) = \langle G; i_0; q_i(i_0), i \in E; x_v(i_0), v \in V \rangle$ will be called layer $i_0$. For each true vertex

$$\sum_{v \in V^{+}(i)} x_v(i_0) - \sum_{v \in V^{-}(i)} x_v(i_0) = q_i(i_0), \qquad i \in E \tag{1}$$

For the vertex $i_0$ will be fair.

$$q_{i_0}(i_0) = -\sum_{i \in E \setminus \{i_0\}} q_i(i_0) \tag{2}$$

The correlation (1) is the first rule of the Kirchhoff network.

We denote by $X_v = \sum_{i_0 \in E} x_v(i_0)$, that this is the total flow that goes through the arc $v$.

From (1) is next $\sum_{v \in V^{+}(i)} X_v - \sum_{v \in V^{-}(i)} X_v = \sum_{i_0 \in E} q_i(i_0), \qquad i \in E \qquad (3)$

Without loss of generality, we will assume that each vertex forms a layer, if for some vertex $i_0$ it does not exist, this means that $q_i(i_0)=0, i \in E$. In the urban transport network (1), (2) are made for all $i_0 \in E$. The viability of (3) does not follow the validity of (1).

**Observation** Single-layer transport systems are of great practical importance. The simplest examples of such tasks are the tasks of entering an institution, evacuating buildings, stadiums, etc.

## 4 Analysis of a single layer system

### 4.1 Find the allowed initial flows using the maximum flow problem in the network

Since only one layer is considered, the index of layer $i_0$ will be ignored. The idea of the equilibrium search algorithm is to look for permissible initial flows in the network and then transform them into a state of equilibrium. Given that each arc has a limited bandwidth, it is possible to verify the existence of valid flows using the maximum flow problem and its solution using the Ford-Fulkerson algorithm.

In the problem of maximum flow, the flow passes from the first initial vertex to the end. All arcs have a specified bandwidth. To reduce the problem to this form, we add two fictitious vertices *ii* and *kk*. We connect *ii* to the source of the flow $i_0$. To this, the bandwidth is equal to $-q_{i_0}(i_0)$. Drain with arcs $q_i(i) > 0$ with vertex *kk*. For these arcs, the performance is the same, respectively $q_i(i)$. We get the problem of the maximum flow in the standard form; we use any of the known algorithms to solve it. If it turns out that the maximum flow is smaller $q_{i_0}(i_0)$, then the initial problem is a layer and, consequently, the problem has no solution. In this case, the minimum cut is outside the additional arcs.

If it turns out that the maximum flow is equal $q_{i_0}(i_0)$, we obtain an admissible flow, which we transfer by invariant transformations to a state of equilibrium.

### 4.2. Invariant transformation of layer flows.

Consider an arbitrary cycle $C$. We define an arbitrary direction of the path, which coincides with the direction of some arc of the cycle $u$. Build the characteristic function:

$$Sign_u(v) = \begin{cases} 0, \text{si } v \notin C \\ +1, v \in C, \text{the direction of the arc coincides with the transverse direction of the cycle,} \\ -1, v \in C, \text{the direction of the arc is opposite to the transverse direction of the cycle} \end{cases}$$

Let $x_v$, $v \in V$ satisfy the relations (1). Take an arbitrary number $\theta$, for all $v \in V$ we put $\overline{x_v} = x_v + sign^u_C(v)\theta$, that is, for cyclic arcs whose direction of which coincides with the direction of the release, $x_v$ is added to the flow value $\theta$, for cyclic arcs whose address is opposite to the direction of derivation, subtract $\theta$ from the flow value $x_v$. Then $\overline{x_v}, v \in V$ satisfies the relation (1).

**4.3. The second Kirchhoff rule for road traffic.**

Consider a single transport flow layer from the source with the number $i_0$, to store the number $j_0$. Suppose that in this movement the flows diverge at the vertex *i* and converge at the vertex *j*. Let some streams go along the arcs that satisfy the First Rule of Kirchhoff. The ways to deliver this flow from *i* to *j* are indicated by *P1* and *P2*. Suppose that the time *t1* of movement along the path 1 turned out to be longer than the time *t2* of movement along the path 2. Then, part of the flow will change to the second path. There will be an increase in the flow of path 2, and consequently the time of this path will increase. At the same time, the value of the flow along the first path decreases and, consequently, the time of movement along the first path decreases. The commutation will stop when the equality *t1 = t2* is maintained, or *t1-t2 = 0*. In general, equality $\sum_{v\in V} sign_C^u(v)(\tau_v(x_v)) = 0$ must be met, and in the theory of hydraulic networks it is known as Kirchhoff's second rule. Which says: In the equilibrium state, the sum of the change in the flow time of the flow throughout the cycle is zero.

With arbitrary threads, this equality is not met. Using the invariant transformation of flows, we can write as $NB_u(\theta) = \sum_{v\in V} sign_C^u(v)\left(\tau_v\left(x_v + sign_C^u(v)\theta\right)\right)$. The task of linking the cycle is to define such $\theta$, that $NB_u(\theta) = 0$.

**4.4. Construction of the nucleus, the fundamental cycle system.**

It is known that if the Kirchhoff rule is executed in a system of fundamental cycles, then it runs in any cycle of the graph. A skeleton is an arbitrary tree whose vertices coincide with the vertices of the original graph *G*, an example of a skeleton is given. The tree can be constructed by any algorithm, for example, the construction of the shortest route tree by the Dijkstra algorithm. The shortest routes can be taken in the direction of the length of the route to the root. The arcs outside the tree are called chords. The fundamental cycle consists of wooden ropes and bows. *C (u)* is the cycle formed by the chord *u*. For each cycle, we establish the direction of the walk, which coincides with the direction of this chord.

Building a loop for a function $sign_u(v)$ is easy.

**4.5. Calculation of the limits of the function $NB_u(\theta)$ change of argument.**

We divide $NB_u(\theta)$ into three parts: $NB_u(\theta) = NB_u^{\ 0}(\theta) + NB_u^{\ +}(\theta) + NB_u^{\ -}(\theta)$. The first part $NB_u^{\ 0}(\theta)$ consists of the terms with *v* for which $sign_u(v) = 0$. The second part $NB_u^{\ +}(\theta)$ includes the terms with *v*, for which $sign_u(v) = 1$, the third part $NB_u^{\ -}(\theta)$ includes the terms with *v*, for which $sign_u(v) = -1$.

1. Obviously $NB_u^{\ 0}(\theta) = 0$.

2. $NB_u^{\ +}(\theta) = \sum_{v\in V,\ sign_u(v)=1} \tau_v(x_v + \theta)$. Just as $\tau_v(x\ ) > 0$ and increasing *x* (see fig 4), so $NB_u^{\ +}(\theta) > 0$ increases by $\theta$. For all $v \in V, sign_u(v) = 1$ executed $0 \le (x_v + \theta) \le x_{\max}$, o $-x_v \le \theta \le x_{\max} - x_v$. Then, we get that

$$\theta \in [\underline{\theta^+}, \overline{\theta^+}], \tag{4}$$

where $\underline{\theta^+} = \max_{v \in V,\, sign_u(v)=1} (-x_v)$, $\overline{\theta^+} = \min_{v \in V,\, sign_u(v)=1} (x_{\max} - x_v)$

3. As well as 2, we obtain that $NB_u^-(\theta)$ increases.

$$\theta \in [\underline{\theta^-}, \overline{\theta^-}], \tag{5}$$

where $\underline{\theta^-} = \max_{v \in V,\, sign_u(v)=-1} (x_v - x_{\max})$, $\overline{\theta^-} = \min_{v \in V,\, sign_u(v)=-1} x_v$

Of the cases considered 1-3 and of the formulas. (4), (5) we have that the function $NB_u(\theta)$ increases, and defines the segment $\theta \in [\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} = \max(\underline{\theta^-}, \underline{\theta^+})$, $\underline{\theta} = \min(\overline{\theta^-}, \overline{\theta^+})$.

**5. Algorithms to find the equilibrium state of the urban transport system.**

These constructions allow the use of algorithms such as the sequential link of cycles to find the equilibrium state of a layer. For example, we are looking for an arc for which $NB_u(0) > \varepsilon$ (a sufficiently small number), if this arc was not found, we stopped linking the layer, for this arc we solve the problem $NB_u(\theta) = 0$ and proceed to execute the algorithm again. For multilayer systems, the search for an arc must be made on all layers and, consequently, within the layer.

**Conclusion.**

Currently, systems for decision making (SPID, Public Digital Identity System) are being developed to analyse the operation of urbanised systems. The unifying name is "Smart City". In the "Smart City" the SPID network [1], [6] is included for the following systems:

- cold water supply,
- hot water supply,
- gas supply,
- sewer system,
- heat supply,
- roads,
- evacuation.
- complex housing, businesses, shops, theatres, stadiums, etc.

SPID "Smart City" is organised as a simulation system. The applied methods of decomposition and composition of tasks allow to analyse urbanised territories of unlimited dimension. Since July 2016, SPID is accepted and recognised by all the EU states members [7]

Experience in problem solving gives reasons for discussing the effectiveness of the proposed approach. Thus, for example, the solution of problems of the distribution of the flow of the urban networks of water supply, the heating with a dimension of approximately 2,000 peaks and about 1,500 arcs for approximately 15 to 30 seconds in personal computers of the consumers suggests that the distribution of the flow in the road networks can be completed in a reasonable time. The proposed algorithms allow the use of methods of their parallelization, which offers wide possibilities of use of modern multiprocessor computer systems.

The area of regional economy, where the theory will be applied to solve these problems should be prepared for projects of integrated development of territories. The study of the regional economy it is important because it helps us to understand the prerequisites for the economic development of the regions, including geographical location, natural resources, demographic, production potential, regional production structure, social sphere, system of settlement and property, mechanism of management of the regional economy, economic relations of the region, etc.

A good road system or a good distribution of vehicular flow (ideally the combination of both elements) generates an optimum well-being in the population and increases production and economic development making the city or nation more competitive. In addition, it increases investment both internally and externally, causing a healthy economy to develop. Assessing the impact of supply and quality of road infrastructure on economic performance is also a complex area of research with potentially important implications on the international infrastructure lending strategy for developing countries [9].

Therefore, it is of vital importance to establish methodologies for the studies, the intervention and the conservation of the infrastructure in order to provide an efficient, comfortable and safe vehicular traffic.

The road infrastructure and communications plus technology make a city “smart”.

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